Communications on Applied Mathematics and Computation
https://doi.org/10.1007/s42967-023-00346-x

ORIGINAL PAPER

Numerical Stability and Convergence for Delay Space-Fractional Fisher Equations with Mixed Boundary Conditions in Two Dimensions

Jing Chen[1] · **Qi Wang**[1]

Received: 26 June 2023 / Revised: 14 September 2023 / Accepted: 23 October 2023

Abstract

In this paper, for generalized two-dimensional delay space-fractional Fisher equations with mixed boundary conditions, we present the stability and convergence computed by a novel numerical method. The unconditional stability of analytic solutions is first derived. Next, we have established the linear θ-method with the Grünwald-Letnikov operator, which has the first-order accuracy in spatial dimensions. Moreover, approaches involved error estimations and inequality reductions are utilized to prove the stability and convergence of numerical solutions under different values of θ. Eventually, we implement a numerical experiment to validate theoretical conclusions, where the interaction impacts of fractional derivatives have been further analyzed by applying two different harmonic operators.

Keywords Space-fractional delay Fisher equation · Grünwald-Letnikov operator · Linear θ-method · Stability · Convergence

Mathematics Subject Classification 65M12 · 35R11 · 47A58

1 Introduction

In last few decades, fractional calculus growing considerable attention has been widely applied in numerous fields of sciences to precisely describe many natural phenomena and dynamic processes. The interested readers can read [2, 28, 30, 33] for more details.

The generalized integral order Fisher equations have cast researchers' eyes due to their broader practicality in various fields, such as dynamics of propagation [34], population estimation [24], epidemic diseases and bacteria [9, 17, 19]. Nevertheless, there are fewer investigations on fractional counterparts. With the utilization of a nonlinear time-fractional

✉ Qi Wang
bmwzwq@126.com

Jing Chen
cjing220701@163.com

1 School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510006, Guangdong, China

Published online: 05 February 2024

Fisher equation, for instance, Wang et al. [37] depicted chemical kinetics via a local meshless method combined with an explicit difference scheme.

In general, the analytic solutions for numerous partial differential equations are difficult to obtain [23, 41]. Therefore, we should redirect that spotlight on qualitative properties of numerical solutions concerning such equations. Zhang and Li [42] proposed linear θ-methods with two kinds of difference operators to study the unique solvability, asymptotic stability, and convergence of space-fractional delayed diffusion equations. Zaky et al. [40] employed a Legendre-based Galerkin spectral method and $L1$-type approximations to explore nonlinear time-space fractional diffusion-reaction equations on both uniform and graded meshes. Hendy et al. [15] constructed a novel numerical technique for solving a nonlinear multi-term time-space fractional delayed differential equation, where the forth-order difference approximation was derived for spatial derivatives using the weighted average of the shifted Grünwald formulae. Due to some macroscopic behaviors of materials, a new Caputo fractional derivative via fixed point theorem was applied to study the nonlinear Fisher's reaction-diffusion equation in Abdon's research [4]. Demir et al. in [11] presented a residual power series method to approximate a time-fractional Fisher equation with a tiny delay. Majeed et al. [25] further applied an efficient cubic B-spline collocation scheme and obtained the unconditional stability. More discrete methods can be found in [12, 31, 43].

Apart from above-mentioned interpretations in the one-dimensional space, the extension of theoretical investigations on multiple dimensions plays a significant role in real-world applications. For a delay fractional partial hyperbolic differential system in two dimensions, Arfaoui and Makhlouf [3] established certain conditions of the stability and illustrated numerical simulations by applying the fix-point approach. Zhu et al. [45] studied the uniqueness of weak solutions, the stability and convergence in the L_2-norm for Riesz space-fractional Fisher equations in two dimensions, where the model is fully discretized by the linearized Crank-Nicolson method. Based on two different time discretization schemes, Oruç [27] presented a more accurate and reliable collocation method with Chebyshev wavelets to calculate two-space dimensional extended Fisher equations. For the dynamics of a cell sheet wound closure, Achouri et al. [1] investigated some numerical properties of the two-dimensional Fisher equation with mixed boundary conditions using energy functionals in the Hilbert space together with the nonlinear Crank-Nicolson scheme. For more details on other numerical methods to solve such equations in two dimensions, the readers can refer to [10, 22, 39] and references therein.

To the authors' best knowledge, when exploring many practical problems related to differential equations, the delayed term is always introduced since we cannot avoid the stochastic influence brought by past states. Even though there are already numerous researches to solve different classes of fractional differential equations with delay, there is less work on the numerical analysis of space-fractional Fisher equations along with a time delay and complex boundary conditions in two dimensions. So in this research, associating mixed boundary conditions with constraints, the generalized two-dimensional space-fractional Fisher equation with a delayed term is considered as follows:

$$\begin{cases} \partial_t u(x,y,t) = \kappa \partial_x^\alpha u(x,y,t) + \gamma \partial_y^\beta u(x,y,t) + \varpi u(x,y,t-\tau)(1-u(x,y,t-\tau)) \\ \qquad\qquad + f(x,y,t),\ \varpi \in \mathbb{R}^*, (x,y,t) \in (0,x_{\rm R}) \times (0,y_{\rm R}) \times (0,T], \\ u(0,y,t) = 0,\ \dfrac{\partial}{\partial x} u(x_{\rm R},y,t) = 0,\ y \in [0,y_{\rm R}], t > 0, \\ u(x,0,t) = 0,\ \dfrac{\partial}{\partial y} u(x,y_{\rm R},t) = 0,\ x \in [0,x_{\rm R}], t > 0, \\ u(x,y,t) = \varphi_0(x,y,t), -\tau \leqslant t \leqslant 0,\ (x,y) \in \bar{I}_x \times \bar{I}_y, \end{cases} \tag{1}$$

where $t > 0, x \in I_x = (0,x_{\rm R}), y \in I_y = (0,y_{\rm R})$, $\kappa, \gamma > 0$ represent the diffusion coefficients, and $\tau > 0$ denotes the time-delay. Particularly, $\partial_x^\alpha u(x,y,t)$ and $\partial_y^\beta u(x,y,t)$ are the Riesz space-fractional derivatives with respect to x and y, respectively, where $\alpha, \beta \in (1,2]$. Supposing $u(x,\ y,\ t)$ is compactly supported on the open interval $I_x \times I_y$, then the definitions regarding Riesz fractional derivatives are enumerated as follows:

$$\begin{cases} \partial_x^\alpha u(x,y,t) = -\dfrac{1}{2\cos(\alpha\pi/2)}\Big({}_0D_x^\alpha u(x,y,t) + {}_xD_{x_{\rm R}}^\alpha u(x,y,t)\Big), \\ \partial_y^\beta u(x,y,t) = -\dfrac{1}{2\cos(\beta\pi/2)}\Big({}_0D_y^\beta u(x,y,t) + {}_yD_{y_{\rm R}}^\beta u(x,y,t)\Big), \end{cases}$$

where ${}_0D_x^\alpha, {}_xD_{x_{\rm R}}^\alpha$ represent two-sides Riemann-Liouville fractional derivative operators in [44], respectively. Their integrated patterns are specifically defined by

$$\begin{cases} {}_0D_x^\alpha u(x,y,t) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{\partial^2}{\partial x^2} \displaystyle\int_0^x u(\xi,y,t)(x-\xi)^{1-\alpha} {\rm d}\xi, \\ {}_xD_{x_{\rm R}}^\alpha u(x,y,t) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{\partial^2}{\partial x^2} \displaystyle\int_x^{x_{\rm R}} u(\xi,y,t)(\xi-x)^{1-\alpha} {\rm d}\xi, \end{cases}$$

where $\Gamma(\cdot) = \int_0^\infty t^{(\cdot-1)} {\rm e}^{-t} {\rm d}t\ (x,y>0)$ denote the Gamma functions referred in [32], and ${}_0D_y^\beta, {}_yD_{y_{\rm R}}^\beta$ are analogously defined as well.

For convenience, we define a nonlinear-reaction term as follows:

$$f(x,y,t,u(x,y,t-\tau)) \triangleq \varpi u(x,y,t-\tau)(1-u(x,y,t-\tau)) + f(x,y,t), \tag{2}$$

and several assumptions should be guaranteed throughout the full paper,

(A1) the analytic and numerical solutions of (1) are sufficiently smooth;
(A2) the continuous function $f(x,y,t,u(x,y,t-\tau))$ satisfies the Lipschitz condition

$$|f(x,y,t,u(x,y,t-\tau)) - f(x,y,t,\bar{u}(x,y,t-\tau))| \leqslant L|u-\bar{u}|, \tag{3}$$

where $(x,y,t) \in I_x \times I_y \times (0,T)$ and L denotes the Lipschitz constant.

On the selection of the numerical method for Problem (1), the linear θ-method is applied to discrete the temporal derivative. The spatial approximation of Riesz fractional derivatives is based on Grünwald-Letnikov operators with the first-order accuracy [26]. Together with our contributions and extensions in this research, numerical investigations with more complicated boundary conditions have been analyzed. Most importantly, it is a remarkable fact that we mentioned the interaction impacts of fractional derivatives computed by two kinds of approximation difference operators when verifying the theoretical effectiveness.

The organization of this paper is as follows. In Sect. 2, the unconditional stability concerning analytic solutions of Problem (1) is studied. We propose the linear θ-method with Grünwald-Letnikov operators to discrete the original equation in Sect. 3. Further analysis involved the stability, local truncation error, and convergence of the numerical scheme are also investigated. In Sect. 4, we present several numerical evidences to confirm our theoretical results. Eventually, some concluding remarks are briefly provided in Sect. 5.

2 Stability of Analytic Solutions

In this section, a stability condition for analytic solutions of Problem (1) is essentially investigated by considering a homogeneous case.

Theorem 1 *The analytic solutions of Problem* (1) *are unconditionally stable for any positive constants* κ, γ.

Proof In order to obtain the stability of analytic solutions, the characteristic function plays a significant role. The following two cases are specifically considered.

Case 1 $\alpha = \beta = 2$. We define a characteristic equation for the homogeneous problem $\partial_t u(x,y,t) = \kappa \partial_x^\alpha u(x,y,t) + \gamma \partial_y^\beta u(x,y,t)$ as follows:

$$\lambda z(x,y) - \kappa \frac{\partial^2 z(x,y)}{\partial x^2} - \gamma \frac{\partial^2 z(x,y)}{\partial y^2} = 0, \ z \in \mathcal{D} \setminus \{(0,0)\}, \tag{4}$$

where $\mathcal{D} = \{u\colon\ u \in C^2(I_x \times I_{\mathrm{R}}) \cap C(\bar{I}_x \times \bar{I}_y)\}$ and the set of continuous functions $\mathcal{D}$ satisfies $u(x,0) = \partial_x u(x_{\mathrm{R}}, y) = 0, u(0,y) = \partial_y u(x, y_{\mathrm{R}}) = 0$.

According to [29], the method of separation of variables is commonly applied. It is worth noting that the eigenvalue problem in two dimensions

$$\begin{cases} \phi''(x)+\mu\phi(x) = 0, \phi(0) = \phi'\left(x_{\mathrm{R}}\right) = 0, x \in \left(0, x_{\mathrm{R}}\right), \\ \psi''(y)+\lambda\psi(y) = \mu\psi(y), \psi(0) = \psi'\left(y_{\mathrm{R}}\right) = 0, y \in \left(0, y_{\mathrm{R}}\right) \end{cases} \tag{5}$$

have eigenvalues $\left\{ \left(\frac{m\pi}{2x_{\mathrm{R}}}\right)^2, \left(\frac{n\pi}{2y_{\mathrm{R}}}\right)^2 \right\}_{m,n=1}^{\infty}$ with eigenfunctions

$$\phi_m(x) = \cos\left(\frac{m\pi x}{2x_{\mathrm{R}}}\right), \psi_n(y) = \cos\left(\frac{n\pi y}{2y_{\mathrm{R}}}\right),$$

where λ, μ denote the non-zero constants.

Substituting

$$z(x_m, y_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z_{mn} \cos\left(\frac{m\pi x}{2x_{\mathrm{R}}}\right) \cos\left(\frac{n\pi y}{2y_{\mathrm{R}}}\right)$$

into (4), we have the following characteristic equation:

$$f(\lambda_{mn}) \triangleq \lambda_{mn} + \kappa \left(\frac{m\pi}{2x_{\mathrm{R}}}\right)^2 + \gamma \left(\frac{n\pi}{2y_{\mathrm{R}}}\right)^2, \ m, n = 1, 2, \cdots. \tag{6}$$

Admittedly, the analytic solutions are stable if all zeros of $f(\lambda_{nm})$ have negative real parts, while it is unstable if at least one zero has a positive real part.

Denote $\lambda_{mn} = a_{mn} + b_{mn}\mathrm{i}$, where $a_{mn}, b_{mn} \in \mathbb{R}$. Then the separations of its real and imaginary parts give

$$a_{mn} = -\kappa\left(\frac{m^2\pi^2}{4x_{\mathrm{R}}^2}\right) - \gamma\left(\frac{n^2\pi^2}{4y_{\mathrm{R}}^2}\right) < 0, b_{mn} = 0, \tag{7}$$

which implies that the analytic solutions of Problem (1) are unconditionally stable.

Case 2 $\alpha, \beta \in (1, 2)$. Associated with some properties of differential operators from [36, 42], the homogeneous case can be redefined by the Fourier transform

$$u_t(x, y, t) + \mathcal{F}\big(-\partial_x^\alpha u(x, y, t)\big) + \mathcal{F}\Big(-\partial_y^\beta u(x, y, t)\Big) = 0, \tag{8}$$

where $\mathcal{F}\big(-\partial_x^\alpha u(x, y, t)\big) = |m|^\alpha \mathrm{e}^{\tilde{\lambda}_{mn}t}$ and $\mathcal{F}\Big(-\partial_y^\beta u(x, y, t)\Big) = |n|^\beta \mathrm{e}^{\tilde{\lambda}_{mn}t}$.

Similar to (6)–(7), it is apparent to find that all zeros of the characteristic equation $\tilde{f}(\tilde{\lambda}_{mn}) \triangleq \tilde{\lambda}_{mn} + \kappa m^\alpha + \gamma n^\beta$ permanently have negative real parts by separating $\tilde{\lambda}_{mn} = \tilde{a}_{mn} + \tilde{b}_{mn}\mathrm{i}$, which shows the analytic solutions of Problem (1) are unconditionally stable. This completes the proof.

3 Numerical Scheme and Corresponding Analysis

In this section, utilizing the harmonic Grünwald-Letnikov operator for the Riesz space-fractional derivative, we will set up a numerical method with the first-order accuracy in space. Then the stability and convergence of numerical solutions will be discussed.

3.1 Theoretical Derivation of the First-Order Numerical Difference Scheme

At the very start, it is essential to introduce the following Grünwald-Letnikov operators and relevant approximations:

$$\begin{cases} \tilde{\Delta}_{h_x}^\alpha u(x, y, t) = \displaystyle\sum_{l=0}^{i+1} \tilde{g}_l^{(\alpha)} u\big(x_{i-l+1}, y_j, t_k\big) + \sum_{l=0}^{M_x-i+1} \tilde{g}_l^{(\alpha)} u\big(x_{i+l-1}, y_j, t_k\big), \\ \tilde{\Delta}_{h_y}^\beta u(x, y, t) = \displaystyle\sum_{l=0}^{j+1} \tilde{g}_l^{(\beta)} u\big(x_i, y_{j-l+1}, t_k\big) + \sum_{l=0}^{M_y-j+1} \tilde{g}_l^{(\beta)} u\big(x_i, y_{j+l-1}, t_k\big) \end{cases} \tag{9}$$

with

$$\tilde{g}_l^{(\alpha)} = (-1)^l \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-l+1)\Gamma(l+1)}, \tilde{g}_l^{(\beta)} = (-1)^l \frac{\Gamma(\beta+1)}{\Gamma(\beta-l+1)\Gamma(l+1)}. \tag{10}$$

Let $C^{1+\alpha}(\mathbb{R})$ be the space of *Hölder* continuous functions with exponent α, whose counterparts have continuities with the first-order derivative in x-direction. $C^{1+\beta}(\mathbb{R})$ can be analogously defined in y-direction.

Lemma 1 ([26]) *Let* $u(x,\cdot,\cdot)\in C^{1+\alpha}(\mathbb{R})$ *and* $u(\cdot,y,\cdot)\in C^{1+\beta}(\mathbb{R})$. *Then for fixed spacial step-sizes* h_x, h_y

$$-\frac{c_\alpha}{h_x^\alpha}\tilde{\Delta}_{h_x}^\alpha u(x,y,t)=\partial_x^\alpha u(x,y,t)+O(h_x),-\frac{c_\beta}{h_y^\beta}\tilde{\Delta}_{h_y}^\beta u(x,y,t)=\partial_y^\beta u(x,y,t)+O(h_y),$$

where $c_\alpha=(2\cos(\alpha\pi/2))^{-1}$ *and* $c_\beta=(2\cos(\beta\pi/2))^{-1}$.

For the temporal segmentation, we represent $\Omega_{\Delta t}=\{t_k|-m\leqslant k\leqslant N\}$ as a consolidated partition on the interval $[-\tau,T]$, where $t_k=k\Delta t\ (k=-m,-m+1,\cdots,0,\cdots,N)$ with the time step-size $\Delta t=\frac{\tau}{m}$. Denote $\Omega_h=\{(x_i,y_j)|0\leqslant i\leqslant M_x, 0\leqslant j\leqslant M_y\}$ as a uniform mesh on the spatial interval $I_x\times I_y$, where $x_i=ih_x$ for $i=0,1,\cdots,M_x$, $y_j=jh_y$ for $j=0,1,\cdots,M_y$ with the space step-sizes $h_x=\left(\frac{x_{\mathrm{R}}-0}{M_x}\right)$ in x-direction and $h_y=\left(\frac{y_{\mathrm{R}}-0}{M_y}\right)$ in y-direction, respectively.

By Lemma 1, we apply the first-order forward differential formula to discrete the left side of Problem (1)

$$\partial_t u(x,y,t)|_{(x_i,y_j,t_k)}=\frac{u_{i,j}^{k+1}-u_{i,j}^k}{\Delta t}, \tag{11}$$

and by utilizing the linear θ-method, we can get the following finite difference scheme:

$$\begin{aligned}\kappa\partial_x^\alpha u|_{(x_i,y_j,t_k)}=&-\frac{\kappa c_\alpha}{h_x^\alpha}\theta\left(\sum_{l=0}^{i+1}\tilde{g}_l^{(\alpha)}u_{i-l+1,j}^{k+1}+\sum_{l=0}^{M_x-i+1}\tilde{g}_l^{(\alpha)}u_{i+l-1,j}^{k+1}\right)\\&-\frac{\kappa c_\alpha}{h_x^\alpha}(1-\theta)\left(\sum_{l=0}^{i+1}\tilde{g}_l^{(\alpha)}u_{i-l+1,j}^{k}+\sum_{l=0}^{M_x-i+1}\tilde{g}_l^{(\alpha)}u_{i+l-1,j}^{k}\right),\end{aligned} \tag{12a}$$

$$\begin{aligned}\gamma\partial_y^\beta u|_{(x_i,y_j,t_k)}=&-\frac{\gamma c_\beta}{h_y^\beta}\theta\left(\sum_{l=0}^{j+1}\tilde{g}_l^{(\beta)}u_{i,j-l+1}^{k+1}+\sum_{l=0}^{M_y-j+1}\tilde{g}_l^{(\beta)}u_{i,j+l-1}^{k+1}\right)\\&-\frac{\gamma c_\beta}{h_y^\beta}(1-\theta)\left(\sum_{l=0}^{j+1}\tilde{g}_l^{(\beta)}u_{i,j-l+1}^{k}+\sum_{l=0}^{M_y-j+1}\tilde{g}_l^{(\beta)}u_{i,j+l-1}^{k}\right),\end{aligned} \tag{12b}$$

and $u_{i,j}^k$ denotes the numerical solution of Problem (1) at point (x_i,y_j,t_k), where $1\leqslant i\leqslant M_x-1, 1\leqslant j\leqslant M_y-1, 1\leqslant k\leqslant N-1$.

Besides, for the defined nonlinear-reaction term (2), we have

$$\begin{aligned}f(x,y,t,u(x,y,t-\tau))|_{(x_i,y_j,t_k)}=&\,\theta f\big(x_i,y_j,t_{k+1},u\big(x_i,y_j,t_{k+1-m}\big)\big)\\&+(1-\theta)f\big(x_i,y_j,t_k,u\big(x_i,y_j,t_{k-m}\big)\big).\end{aligned} \tag{13}$$

When discretizing initial and mixed boundary conditions, the limit definitions of $\partial_x u(x_{\mathrm{R}},y,t)=\partial_y u(x,y_{\mathrm{R}},t)=0$ are indispensable elements, which satisfy

$$\lim_{h_x \to 0} \frac{u(x_R, y, t) - u(x_R - 2h_x, y, t)}{2h_x} = 0, \lim_{h_y \to 0} \frac{u(x, y_R, t) - u(x, y_R - 2h_y, t)}{2h_y} = 0,$$

then we can obtain

$$\begin{cases} u_{0,j}^k = \dfrac{u_{M_x,j}^k - u_{M_x-2,j}^k}{2h_x} = 0, u_{i,0}^k = \dfrac{u_{i,M_y}^k - u_{i,M_y-2}^k}{2h_y} = 0, \\ u_{i,j}^k = \varphi_0\left(x_i, y_j, t_k\right), k = -m, -m+1, \cdots, 0. \end{cases} \tag{14}$$

For further analysis, define some space-fractional partial difference operators

$$\begin{cases} \tilde{\delta}_x^\alpha u_{i,j}^k = -\dfrac{\kappa c_\alpha}{h_x^\alpha}\left(\displaystyle\sum_{l=0}^{i+1} \tilde{g}_l^{(\alpha)} u_{i-l+1,j}^k + \sum_{l=0}^{M_x-i+1} \tilde{g}_l^{(\alpha)} u_{i+l-1,j}^k\right), \\ \tilde{\delta}_y^\beta u_{i,j}^k = -\dfrac{\gamma c_\beta}{h_y^\beta}\left(\displaystyle\sum_{l=0}^{j+1} \tilde{g}_l^{(\beta)} u_{i,j-l+1}^k + \sum_{l=0}^{M_y-j+1} \tilde{g}_l^{(\beta)} u_{i,j+l-1}^k\right), \end{cases}$$

the finite difference scheme (11)–(13) can be subsequently rewritten as

$$\begin{aligned} \left(1 - \Delta t\theta\tilde{\delta}_x^\alpha - \Delta t\theta\tilde{\delta}_y^\beta\right)u_{i,j}^{k+1} &= \left(1 + \Delta t(1-\theta)\tilde{\delta}_x^\alpha + \Delta t(1-\theta)\tilde{\delta}_y^\beta\right)u_{i,j}^k \\ &\quad + \Delta t\left(\theta f\left(x_i, y_j, t_{k+1}, u_{i,j}^{k+1-m}\right) + (1-\theta)f\left(x_i, y_j, t_k, u_{i,j}^{k-m}\right)\right). \end{aligned} \tag{15}$$

Owing to the complexities of computations for high-dimensional problems, Alternating Direction Implicit Methods (ADIMs) have been efficiently applied. Then we will integrate the numerical scheme (15) into a novel operator form as follows:

$$\begin{aligned} \left(1 - \Delta t\theta\tilde{\delta}_x^\alpha\right)\left(1 - \Delta t\theta\tilde{\delta}_y^\beta\right)u_{i,j}^{k+1} &= \left(1 + \Delta t(1-\theta)\tilde{\delta}_x^\alpha\right)\left(1 + \Delta t(1-\theta)\tilde{\delta}_y^\beta\right)u_{i,j}^k \\ &\quad + \Delta t\left(\theta f\left(x_i, y_j, t_{k+1}, u_{i,j}^{k+1-m}\right) + (1-\theta)f\left(x_i, y_j, t_k, u_{i,j}^{k-m}\right)\right), \end{aligned} \tag{16}$$

where the extra perturbation error $(\Delta t\theta)^2(\tilde{\delta}_x^\alpha\tilde{\delta}_y^\beta)u_{i,j}^{k+1} - (\Delta t(1-\theta))^2(\tilde{\delta}_x^\alpha\tilde{\delta}_y^\beta)u_{i,j}^k$ can be negligible in comparison with the local truncation error $R_{i,j}^k(\theta)$.

Additionally, we introduce an intermediate variable $u_{i,j}^*$, the numerical scheme (16) will be decomposed into two independent equations, namely

$$\begin{cases} \left(1 - \Delta t\theta\tilde{\delta}_x^\alpha\right)u_{i,j}^* = \left(1 + \Delta t(1-\theta)\tilde{\delta}_x^\alpha\right)u_{i,j}^k + \Delta t\theta f\left(x_i, y_j.t_k, u_{i,j}^{k+1-m}\right) \\ \qquad\qquad + \Delta t(1-\theta)f\left(x_i, y_j.t_k, u_{i,j}^{k-m}\right), \\ \left(1 - \Delta t\theta\tilde{\delta}_y^\beta\right)u_{i,j}^{k+1} = \left(1 + \Delta t(1-\theta)\tilde{\delta}_y^\beta\right)u_{i,j}^*, \end{cases} \tag{17}$$

where the relevant perturbation error concerning f is analogously ignored.

Denote $\tilde{a}_x = c_\alpha a_x = \kappa c_\alpha \Delta t/h_x^\alpha$ and $\tilde{b}_y = c_\beta b_y = \gamma c_\beta \Delta t/h_y^\beta$, then we rewrite the linear θ-method with Grünwald-Letnikov operators (14) and (17) into the following matrix form:

$$\begin{cases} \left(I + \tilde{a}_x\theta\tilde{A}\right)U_j^* = \left(I - \tilde{a}_x(1-\theta)\tilde{A}\right)U_j^k + \Delta t\theta F_j^{k+1} + \Delta t(1-\theta)F_j^k, \\ \left(I + \tilde{b}_y\theta\tilde{B}\right)U_i^{k+1} = \left(I - \tilde{b}_y(1-\theta)\tilde{B}\right)U_i^*, \end{cases} \tag{18}$$

where the corresponding coefficient matrices are defined by

$$\tilde{A} = \begin{pmatrix} 2\tilde{g}_1^{(\alpha)} & \tilde{g}_0^{(\alpha)} + \tilde{g}_2^{(\alpha)} & \cdots & \tilde{g}_{M_x-2}^{(\alpha)} + \tilde{g}_{M_x}^{(\alpha)} & \tilde{g}_{M_x-1}^{(\alpha)} \\ \tilde{g}_0^{(\alpha)} + \tilde{g}_2^{(\alpha)} & 2\tilde{g}_1^{(\alpha)} & \cdots & \tilde{g}_{M_x-3}^{(\alpha)} + \tilde{g}_{M_x-1}^{(\alpha)} & \tilde{g}_{M_x-2}^{(\alpha)} \\ \vdots & \vdots & & \vdots & \vdots \\ \tilde{g}_{M_x-2}^{(\alpha)} & \tilde{g}_{M_x-3}^{(\alpha)} & \cdots & 2\tilde{g}_1^{(\alpha)} + \tilde{g}_3^{(\alpha)} & \tilde{g}_0^{(\alpha)} + \tilde{g}_2^{(\alpha)} \\ \tilde{g}_{M_x-1}^{(\alpha)} & \tilde{g}_{M_x-2}^{(\alpha)} & \cdots & 2\left(\tilde{g}_0^{(\alpha)} + \tilde{g}_2^{(\alpha)}\right) & 2\tilde{g}_1^{(\alpha)} \end{pmatrix}, \tag{19a}$$

$$\tilde{B} = \begin{pmatrix} 2\tilde{g}_1^{(\beta)} & \tilde{g}_0^{(\beta)} + \tilde{g}_2^{(\beta)} & \cdots & \tilde{g}_{M_y-2}^{(\beta)} + \tilde{g}_{M_y}^{(\beta)} & \tilde{g}_{M_y-1}^{(\beta)} \\ \tilde{g}_0^{(\alpha)} + \tilde{g}_2^{(\alpha)} & 2\tilde{g}_1^{(\beta)} & \cdots & \tilde{g}_{M_y-3}^{(\beta)} + \tilde{g}_{M_y-1}^{(\beta)} & \tilde{g}_{M_y-2}^{(\beta)} \\ \vdots & \vdots & & \vdots & \vdots \\ \tilde{g}_{M_y-2}^{(\beta)} & \tilde{g}_{M_y-3}^{(\beta)} & \cdots & 2\tilde{g}_1^{(\beta)} + \tilde{g}_3^{(\beta)} & \tilde{g}_0^{(\beta)} + \tilde{g}_2^{(\beta)} \\ \tilde{g}_{M_y-1}^{(\beta)} & \tilde{g}_{M_y-2}^{(\beta)} & \cdots & 2\left(\tilde{g}_0^{(\beta)} + \tilde{g}_2^{(\beta)}\right) & 2\tilde{g}_1^{(\beta)} \end{pmatrix}, \tag{19b}$$

and

$$\begin{aligned} U_j^* &= \left(u_{1,j}^*, u_{2,j}^*, \cdots, u_{M_x-1,j}^*\right)^{\mathrm{T}}, U_j^k = \left(u_{1,j}^k, u_{2,j}^k, \cdots, u_{M_x-1,j}^k\right)^{\mathrm{T}}, \\ F_j^k &= \left(f(x_1, y_j, t_k, u_{1,j}^{k-m}), \cdots, f(x_{M_x-1}, y_j, t_k, u_{M_x-1,j}^{k-m})\right)^{\mathrm{T}}, \\ U_i^* &= \left(u_{i,1}^*, u_{i,2}^*, \cdots, u_{i,M_y-1}^*\right)^{\mathrm{T}}, U_i^{k+1} = \left(u_{i,1}^{k+1}, u_{i,2}^{k+1}, \cdots, u_{i,M_y-1}^{k+1}\right)^{\mathrm{T}}. \end{aligned}$$

Therefore, we obtain the entire matrix form of (18) as follows:

$$\begin{aligned} \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)U^{k+1} = \left(I - \tilde{a}_x(1-\theta)\tilde{S} - \tilde{b}_y(1-\theta)\tilde{T}\right)U^k + \Delta t\theta F^{k+1} \\ + \Delta t(1-\theta)F^k, \end{aligned} \tag{20}$$

where

$$\begin{aligned} U^k =& \left(u_{1,1}^k, \cdots, u_{M_x-1,1}^k, \cdots, u_{1,M_y-1}^k, \cdots, u_{M_x-1,M_y-1}^k\right)^{\mathrm{T}}, \\ F^k =& (f(x_1, y_1, t_k, u_{1,1}^{k-m}), \cdots, f(x_{M_x-1}, y_1, t_k, u_{M_x-1,1}^{k-m}), \cdots, \\ & f(x_1, y_{M_y-1}, t_k, u_{1,M_y-1}^{k-m}), \cdots, f(x_{M_x-1}, y_{M_y-1}, t_k, u_{M_x-1,M_y-1}^{k-m}))^{\mathrm{T}}. \end{aligned}$$

Moreover, it is of necessity to introduce the formations of $\tilde{S}$ and $\tilde{T}$, whose elements are determined by (19a) and (19b). $\tilde{S}$ is a block diagonal matrix with $(M_y - 1)$ orders, whose blocks are $(M_x - 1) \times (M_x - 1)$ square matrices concerning $\tilde{A}$ in the numerical method, namely, $\tilde{S} = \mathrm{diag}\left(\tilde{A}, \tilde{A}, \cdots, \tilde{A}\right)$. $\tilde{T}$ is an $(M_y - 1) \times (M_y - 1)$ block matrix whose non-zero blocks are diagonal matrices with $(M_x - 1)$ orders. To put it differently, $\tilde{T}_{i,j}$ is a diagonal matrix resulting from $\tilde{B}$, where $\tilde{T}_{i,j} = \mathrm{diag}\left(\tilde{b}_{i,j}, \tilde{b}_{i,j}, \cdots, \tilde{b}_{i,j}\right), \tilde{b}_{i,j} \in \tilde{B}$.

3.2 Some Relevant Properties of Coefficient Matrices

Lemma 2 ([39]) *Let $\tilde{g}_l^{(\alpha)}$ be coefficients of the Grünwald-Letnikov difference approximation (9)–(10). Then*

(a) $\tilde{g}_0^{(\alpha)} = 1, \tilde{g}_1^{(\alpha)} = -\alpha < 0$;

(b) $\tilde{g}_l^{(\alpha)} > 0$, *for all* $l = 0, 2, 3, \cdots$;

(c) $\sum\limits_{l=0}^{\infty} \tilde{g}_l^{(\alpha)} = 0$.

Lemma 3 ([8]) *If the matrix $D = (d_{il})_{m\times m}$ satisfies the follwing conditions*:

$$\sum_{l=1,l\neq i}^{m} \left|d_{il}\right| \leqslant \left|d_{ii}\right| - 1,\ i = 1, 2, \cdots, m, \tag{21}$$

then the norm inequality regarding $X \in \mathbb{R}^m$ holds

$$\|X\|_\infty \leqslant \|DX\|_\infty, \tag{22}$$

where its maximum norm is defined by $\|X\|_\infty = \max\limits_{1\leqslant i\leqslant m}\left|x_i\right|$.

Before the explorations of coefficient matrices $c_\alpha\tilde{S}, c_\beta\tilde{T}$, it is essential to principally investigate the positive properties of $c_\alpha\tilde{A}, c_\beta\tilde{B}$.

First and foremost, associating Lemma 2 with the *Gershgorin's Disk* Theorem [16], we can obtain the upper and lower bounds of $\lambda_{c_\alpha\tilde{A}}$ as follows:

$$\begin{cases} \left|\lambda_{c_\alpha\tilde{A}} - 2c_\alpha\tilde{g}_1^{(\alpha)}\right| \leqslant -c_\alpha\left(\sum\limits_{l=0,l\neq 1}^{i}\tilde{g}_l^{(\alpha)} + \sum\limits_{l=0,l\neq 1}^{M_x-i+1}\tilde{g}_l^{(\alpha)}\right) < -c_\alpha\left(-\tilde{g}_1^{(\alpha)} - \tilde{g}_1^{(\alpha)}\right) \\ \qquad = 2c_\alpha\tilde{g}_1^{(\alpha)}, \text{when } i \neq M_x - 2, \\ \left|\lambda_{c_\alpha\tilde{A}} - 2c_\alpha\tilde{g}_1^{(\alpha)}\right| \leqslant \left|\lambda_{c_\alpha\tilde{A}} - c_\alpha\left(2\tilde{g}_1^{(\alpha)} + \tilde{g}_3^{(\alpha)}\right)\right| \leqslant -c_\alpha\left(\sum\limits_{l=0,l\neq 1}^{M_x-2}\tilde{g}_l^{(\alpha)} + \sum\limits_{l=0,l\neq 1}^{3}\tilde{g}_l^{(\alpha)}\right) \\ \qquad < -c_\alpha\left(-\tilde{g}_1^{(\alpha)} - \tilde{g}_1^{(\alpha)}\right) = 2c_\alpha\tilde{g}_1^{(\alpha)}. \end{cases}$$

Consequently, we have

$$0 < \lambda_{c_\alpha\tilde{A}} < 4c_\alpha\tilde{g}_1^{(\alpha)}, \tag{23}$$

which indicates $c_\alpha\tilde{A}$ is positive definite since the negative properties of c_α and $\tilde{g}_1^{(\alpha)}$. Similarly, we can prove that $c_\beta\tilde{B}$ is positive definite because of $0 < \lambda_{c_\beta\tilde{B}} < 4c_\beta\tilde{g}_1^{(\beta)}$ as well. Furthermore, we apply Matlab built-in command **eig** to numerically calculate the ranges of eigenvalues $\lambda_{c_\alpha\tilde{A}}$ and $\lambda_{c_\beta\tilde{B}}$, listed in Table 1 by different orders α, β.

Based on these discussions mentioned above, we could derive the following theorem concering $c_\alpha\tilde{S}, c_\beta\tilde{T}$.

Theorem 2 *Suppose that $1 < \alpha, \beta \leqslant 2$, then the matrices $c_\alpha\tilde{S}, c_\beta\tilde{T}$ defined by (20) are positive definite. Additionally, the coefficient matrix satisfies*

Table 1 The minimum and maximum eigenvalues of the positive coefficient matrices $c_\alpha\tilde{A}, c_\beta\tilde{B}$

M_x	$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
	min $\lambda_{c_\alpha\tilde{A}}$	max $\lambda_{c_\alpha\tilde{A}}$	min $\lambda_{c_\alpha\tilde{A}}$	max $\lambda_{c_\alpha\tilde{A}}$	min $\lambda_{c_\alpha\tilde{A}}$	max $\lambda_{c_\alpha\tilde{A}}$
10	0.191 248 48	3.876 210 43	0.146 710 56	3.538 612 00	0.121 331 75	3.670 407 90
100	0.004 666 24	3.999 105 07	0.002 480 91	3.645 617 95	0.001 352 10	3.777 741 08
1 000	0.000 253 69	3.999 992 03	0.000 075 77	3.646 440 81	0.000 020 42	3.778 644 56
M_y	$\beta = 1.2$		$\beta = 1.6$		$\beta = 1.8$	
	min $\lambda_{c_\beta\tilde{B}}$	max $\lambda_{c_\beta\tilde{B}}$	min $\lambda_{c_\beta\tilde{B}}$	max λ_B	min $\lambda_{c_\beta\tilde{B}}$	max $\lambda_{c_\beta\tilde{B}}$
50	0.032 038 28	7.427 225 77	0.010 536 74	3.743 575 11	0.006 480 57	3.657 886 96
500	0.002 046 02	7.434 466 36	0.000 279 07	3.747 024 91	0.000 104 61	3.661 370 75
5 000	0.001 181 65	7.434 535 36	0.000 092 56	3.747 057 52	0.000 018 69	3.661 403 89

$$\|X\|_\infty \leqslant \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)X \right\|_\infty. \tag{24}$$

Proof First of all, it is apparent to know that $c_\alpha\tilde{S}, c_\beta\tilde{T}$ are positive definite due to the relationships between $c_\alpha\tilde{A}$ and $c_\alpha\tilde{S}$ as well as $c_\beta\tilde{B}$ and $c_\beta\tilde{T}$.

Moreover, we clearly observe that elements of $c_\alpha\tilde{A}$ and $c_\beta\tilde{B}$ satisfy the following conditions enlightened by Lemma 3:

$$\begin{cases} \displaystyle\sum_{l=1,l\neq i}^{M_x-1} \left|\tilde{a}_x\theta\tilde{a}_{il}\right| = \left|\tilde{a}_x\right|\theta\left(\sum_{l=0,l\neq 1}^{i}\left|\tilde{g}_l^{(\alpha)}\right| + \sum_{l=0,l\neq 1}^{M_x-i+1}\left|\tilde{g}_l^{(\alpha)}\right|\right), \\ \displaystyle\sum_{l=1,l\neq j}^{M_y-1} \left|\tilde{b}_y\theta\tilde{b}_{jl}\right| = \left|\tilde{b}_y\right|\theta\left(\sum_{l=0,l\neq 1}^{j}\left|\tilde{g}_l^{(\beta)}\right| + \sum_{l=0,l\neq 1}^{M_y-j+1}\left|\tilde{g}_l^{(\beta)}\right|\right), \end{cases}$$

which will further illustrate

$$\begin{cases} \displaystyle\sum_{l=1,l\neq i}^{M_x-1} \left|\tilde{a}_x\theta\tilde{a}_{il}\right| \leqslant \begin{cases} a_x\theta\cdot 2c_\alpha\tilde{g}_1^{(\alpha)}, & i\neq M_x-2, \\ \left|\tilde{a}_x\theta\left(2\tilde{g}_1^{(\alpha)}+\tilde{g}_3^{(\alpha)}\right)\right| < a_x\theta\cdot 2c_\alpha\tilde{g}_1^{(\alpha)}, & i = M_x-2, \end{cases} \\ \displaystyle\sum_{l=1,l\neq j}^{M_y-1} \left|\tilde{b}_y\theta\tilde{b}_{jl}\right| \leqslant \begin{cases} b_y\theta\cdot 2c_\beta\tilde{g}_1^{(\beta)}, & j\neq M_y-2, \\ \left|\tilde{b}_y\theta\left(2\tilde{g}_1^{(\beta)}+\tilde{g}_3^{(\beta)}\right)\right| < b_y\theta\cdot 2c_\beta\tilde{g}_1^{(\beta)}, & j = M_y-2. \end{cases} \end{cases}$$

Therefore, the linear combination of norm relations is obviously acquired

$$\|X\|_\infty \leqslant \left\| \left(I + a_x\theta c_\alpha\tilde{A} + b_y\theta c_\beta\tilde{B}\right)X \right\|_\infty, \tag{25}$$

which directly completes the proof because of their similarities between matrices $\tilde{A}, \tilde{B}$ and $\tilde{S}, \tilde{T}$.

3.3 Stability Analysis

To demonstrate the numerical stability, we divide Problem (20) with different values of θ into two circumstances for analysis. Meanwhile, the relevant definition will be first proposed which is fundamental for the following research.

Definition 1 (See, e.g., [16]) Let $A \in \mathbb{R}^{n\times n}$, and the induced ∞-norm be defined by

$$\|A\|_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \left|a_{ij}\right|, \tag{26}$$

where a_{ij} denotes the matrix element of A.

Then define the complex variable polynomial ($z \in \mathbb{C}$),

$$\tilde{P}_{ij}^{\theta}(z) = \tilde{p}_{ij}(z)z^m - \tilde{q}_{ij}(z), \tag{27}$$

where $\tilde{q}_{ij}(z)$ is a zero-polynomial, and

$$\tilde{p}_{ij}(z) = \left(1 + \theta\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)\right)z - \left(1 - (1-\theta)\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)\right).$$

Then supported by the certain **Lemma** which is common knowledge from [22], for any $m \geqslant 1$, a useful lemma is deduced as follows.

Lemma 4 *$\tilde{P}_{ij}^{\theta}(z)$ is a Schur polynomial if and only if the following condition holds when $\theta \in [0, 1/2)$:*

$$a_x + b_y < \frac{2}{\left(\max\limits_{\substack{1\leqslant i\leqslant M_x-1,\\ 1\leqslant j\leqslant M_y-1}} \left\{\lambda_i^{c_\alpha\tilde{S}}, \lambda_j^{c_\beta\tilde{T}}\right\}\right)(1-2\theta)}, \tag{28}$$

where $\lambda_i^{c_\alpha\tilde{S}}, \lambda_j^{c_\beta\tilde{T}}$ are closely related to fractional derivatives α, β, respectively.

Proof To begin with, we need denote the maximum of $\lambda_i^{c_\alpha\tilde{S}}$ and $\lambda_j^{c_\beta\tilde{T}}$ as λ.

Firstly, we consider the sufficiency. According to $\tilde{p}_{ij}(z) = 0$, we obtain

$$|z| = \left|1 - \frac{a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}}{1 + \theta\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)}\right|. \tag{29}$$

Combining Condition (28) with positive properties of a_x and b_y, we can easily know that $0 < a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}} < (a_x + b_y)\lambda < 2/(1-2\theta)$, hence the corresponding inequalities are satisfied as follows:

$$0 < \frac{a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}}{1+\theta\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)} = \frac{1}{\theta + 1/\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)} < \frac{1}{\theta + (1-2\theta)/2} = 2,$$

which implies that $|z| < 1$ holds deduced by $\tilde{p}_{ij}(z) = 0$.

Let $\tilde{p}_{ij}(z) \triangleq \tilde{w}$. Then we set up a complex variable function

$$z = \frac{\tilde{w} + 1 - (1-\theta)\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)}{1+\theta\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)}, \tag{30}$$

where $\tilde{w} = \tilde{x} + \tilde{y}\mathrm{i}$ $(\tilde{x}, \tilde{y} \in \mathbb{R})$ denotes an imaginary number.

Similar to the derivation processes in [38], the minimum of $|\tilde{w}|$ can be simplified as follows:

$$\min_{|z|=1, z\in\mathbb{C}} |w| = \min\left\{ a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}, \left| -2 + (1-2\theta)\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)\right| \right\}.$$

Case 1 Suppose that $\min\limits_{|z|=1, z\in\mathbb{C}} \left|\tilde{p}_{ij}(z)\right| = a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}$.

With positive properties of eigenvalues computed by $c_\alpha\tilde{S}, c_\beta\tilde{T}$, and coefficients a_x, b_y, we obviously discover that $\left|\tilde{p}_{ij}(z)\right| \geqslant a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}} > 0$ holds.

Case 2 Suppose that $\min\limits_{|z|=1, z\in\mathbb{C}} \left|\tilde{p}_{ij}(z)\right| = \left| -2 + (1-2\theta)\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)\right|$.

By Condition (28), we have

$$0 < \left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)(1-2\theta) < \left(a_x + b_y\right)\lambda(1-2\theta) < 2,$$

which further implies that

$$\left|\tilde{p}_{ij}(z)\right| \geqslant \left| -2 + (1-2\theta)\left(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\right)\right| > 0.$$

So for any $|z| = 1, z \in \mathbb{C}$, the non-zero restrictions of $\tilde{p}_{ij}(z)z^m - \tilde{q}_{ij}(z)$ and $\left|\tilde{p}_{ij}(z)\right| \geqslant \left|\tilde{q}_{ij}(z)\right|$ are simultaneously satisfied, which conclude that $\tilde{P}_{ij}^{\theta}(z)$ is a Schur polynomial.

In what follows, we proceed to prove the necessity by contradictions.

Case 1 Suppose that $a_x + b_y = \frac{2}{\lambda(1-2\theta)}$. Substituting $z = -1$ and $\lambda_i^{c_\alpha\tilde{S}} = \lambda_j^{c_\beta\tilde{T}} = \lambda$ into (27) when m is even, then we have

$$\tilde{P}_{ij}^{\theta}(-1) = -2 + \left(a_x + b_y\right)\lambda(1-2\theta) = 0,$$

which contradicts the non-zero constraints.

Case 2 Suppose that $a_x + b_y > \frac{2}{\lambda(1-2\theta)}$. From (29) we have

$$0 < \frac{a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}}{1 + \theta\left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)} < \frac{1}{\theta + 1/\lambda\left(a_x + b_y\right)} > \frac{1}{\theta + (1-2\theta)/2} = 2,$$

then there exists a possibility of $|z| > 1$ since $1/\left(\theta + 1/\lambda\left(a_x + b_y\right)\right)$ is an infinity value greater than 2, which contradicts aforementioned discussions.

In general, the necessity and sufficiency are completely proved.

With Lemma 4, we can derive a theorem of stability when $\alpha, \beta \in (1,2]$ and $\theta \in [0, 1/2)$.

Theorem 3 *Accompanied with strictly positive properties of a_x, b_y, the linear θ-method with the Grünwald-Letnikov operator is stable when $\alpha, \beta \in (1,2]$ and $\theta \in [0, 1/2)$ if and only if Condition (28) holds.*

Proof At the very beginning, we need to introduce the numerical solution $\bar{u}_{i,j}^k$ with a tiny perturbation calculated by the proposed finite difference scheme

$$\bar{u}_{i,j}^k = \varphi_0(x_i, y_j, t_k) + \psi_{i,j}^k, -m \leqslant k \leqslant 0, \tag{31}$$

where $\psi_{i,j}^k$ is influenced by the initial condition. Denote

$$\varepsilon^k = \left(\varepsilon_{1,1}^k, \cdots, \varepsilon_{M_x-1,1}^k, \varepsilon_{1,2}^k, \cdots, \varepsilon_{M_x-1,2}^k, \cdots, \varepsilon_{1,M_y-1}^k, \cdots, \varepsilon_{M_x-1,M_y-1}^k\right)^{\mathrm{T}},$$

where $\varepsilon_{i,j}^k = \bar{u}_{i,j}^k - u_{i,j}^k$.

Moreover, it is essential to define an error regarding the nonlinear-reaction term

$$\beta^k = \left(\beta_{1,1}^k, \cdots, \beta_{M_x-1,1}^k, \beta_{1,2}^k, \cdots, \beta_{M_x-1,2}^k, \cdots, \beta_{1,M_y-1}^k, \cdots, \beta_{M_x-1,M_y-1}^k\right)^{\mathrm{T}},$$

where $\beta_{i,j}^k = f\left(x_i, y_j, t_k, \bar{u}_{i,j}^{k-m}\right) - f\left(x_i, y_j, t_k, u_{i,j}^{k-m}\right)$, and we should assure the Lipschitz condition is satisfied as follows:

$$\left|\beta_{i,j}^k\right| \leqslant L_1 \left|\bar{u}_{i,j}^{k-m} - u_{i,j}^{k-m}\right|, L_1 \in \mathbb{R}^*. \tag{32}$$

With the hypothesis said before, we will take

$$\begin{aligned} \varepsilon^{k+1} = &\left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\left(I - \tilde{a}_x(1-\theta)\tilde{S} - \tilde{b}_y(1-\theta)\tilde{T}\right)\varepsilon^k \\ &+ \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\left(\Delta t\theta\beta^{k+1} + \Delta t(1-\theta)\beta^k\right) \end{aligned} \tag{33}$$

into consideration, in other words, we concentrate on

$$\begin{aligned} \|\varepsilon^{k+1}\|_\infty \leqslant &\left\|\left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\left(I - \tilde{a}_x(1-\theta)\tilde{S} - \tilde{b}_y(1-\theta)\tilde{T}\right)\right\|_\infty \|\varepsilon^k\|_\infty \\ &+ \left\|\left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\right\|_\infty \left(\Delta t\theta\|\beta^{k+1}\|_\infty + \Delta t(1-\theta)\|\beta^k\|_\infty\right) \end{aligned}$$

for the error terms β^k, it evidently follows from the Lipschitz condition (32) that

$$\Delta t\theta\|\beta^{k+1}\|_\infty + \Delta t(1-\theta)\|\beta^k\|_\infty \leqslant L_1\Delta t\big(\theta\|\varepsilon^{k+1-m}\|_\infty + (1-\theta)\|\varepsilon^{k-m}\|_\infty\big).$$

Besides, it is necessary to approximate the maximum norm of coefficient matrices, where Lemma 4 plays an essential role for the following work.

Associating investigations of Theorem 2 with eigenvalues' discussions in Sect. 3.2, we can convert this problem into analyzing the maximum of

$$\max_{\substack{1\leqslant i\leqslant M_x-1,\\ 1\leqslant j\leqslant M_y-1}} \left| \frac{1-(1-\theta)\Big(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\Big)}{1+\theta\Big(a_x\lambda_i^{c_\alpha\tilde{S}} + b_y\lambda_j^{c_\beta\tilde{T}}\Big)} \right|, \tag{34}$$

which is equivalent to the norm accompanied with $\|\varepsilon^k\|_\infty$.

By Lemma 4, we can easily derive

$$\left\|\big(I+\tilde{a}_x\theta\tilde{S}+\tilde{b}_y\theta\tilde{T}\big)^{-1}\big(I-\tilde{a}_x(1-\theta)\tilde{S}-\tilde{b}_y(1-\theta)\tilde{T}\big)\right\|_\infty < 1. \tag{35}$$

Otherwise, its norm is a boundless value if Condition (28) is not satisfied.

For further simplifications, suppose that

$$e^k = \max\left\{ \max_{0\leqslant i\leqslant k}\left\{\|\varepsilon^k\|_\infty\right\}, \max_{\substack{1\leqslant i\leqslant M_x-1,1\leqslant j\leqslant M_y-1,\\ -m\leqslant k\leqslant 0}} \left\{\left|\psi_{i,j}^k\right|\right\} \right\},$$

where m is a positive integer greater than 1, then we have

$$e^0 = \max_{\substack{1\leqslant i\leqslant M_x-1,1\leqslant j\leqslant M_y-1,\\ -m\leqslant k\leqslant 0}} \left\{\left|\psi_{i,j}^k\right|\right\}, e^0 \leqslant e^1 \leqslant \cdots \leqslant e^{k-1} \leqslant e^k. \tag{36}$$

Therefore, the error $\|\varepsilon^{k+1}\|_\infty$ can be improved as

$$\begin{aligned} e^{k+1} &\leqslant \left\|\big(I+\tilde{a}_x\theta\tilde{S}+\tilde{b}_y\theta\tilde{T}\big)^{-1}\big(I-\tilde{a}_x(1-\theta)\tilde{S}-\tilde{b}_y(1-\theta)\tilde{T}\big)\right\|_\infty e^k \\ &\quad + \left\|\big(I+\tilde{a}_x\theta\tilde{S}+\tilde{b}_y\theta\tilde{T}\big)^{-1}\right\|_\infty L_1\Delta t\big(\theta e^{k+1-m} + (1-\theta)e^{k-m}\big) \\ &\leqslant e^k + L_1\Delta t(\theta e^k + (1-\theta)e^k) = (1+L_1\Delta t)e^k. \end{aligned} \tag{37}$$

Using multiple iteration processes, the conditional stability with (28) will be properly demonstrated, and we can arrive at the following inequalities:

$$e^{k+1} \leqslant (1+L_1\Delta t)^{k+1}e^0 \leqslant \exp\big((k+1)\Delta t\cdot L_1\big)e^0 \leqslant C\exp(TL_1)e^0, \tag{38}$$

where C represents a positive constant which is dependent on Δt and h_x, h_y.

Under the situation $\theta \in [1/2, 1]$, the unconditional stability will be given below.

Theorem 4 *The linear θ-method with the Grünwald-Letnikov operator is unconditionally stable for $\alpha, \beta \in (1,2], \theta \in [1/2, 1]$.*

Proof At the beginning, we define a continuous function with respect to θ,

$$\tilde{f}(\theta) = -\frac{a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}}{1 + \theta\left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)}, \tag{39}$$

then we observe that $\tilde{f}(\theta)$ is a strictly monotonically increasing function since

$$\tilde{f}'(\theta) = \left(\frac{a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}}{1 + \theta\left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)}\right)^2 > 0.$$

Hence, applying the *Intermediate Value Theorem*, the values of $\tilde{f}$ at its endpoints are simplified as

$$\tilde{f}\left(\frac{1}{2}\right) = \frac{-2\left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)}{2 + \left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)}, \tilde{f}(1) = \frac{-\left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)}{1 + \left(a_x \lambda_i^{c_\alpha \tilde{S}} + b_y \lambda_j^{c_\beta \tilde{T}}\right)},$$

which further implies the maximum of (34) is analogously less than 1 due to

$$-2 < \tilde{f}\left(\frac{1}{2}\right) \leqslant \tilde{f}(\theta) \leqslant \tilde{f}(1) < 0.$$

Similar to the investigations given by Theorem 3, the inequality (35) is unconditionally satisfied, and we can easily know the error reductions under the situation $\theta \in [1/2, 1]$ are identical with (37)–(38).

In conclusion, the unconditional stability when $\theta \in [1/2, 1]$ is completely proved.

3.4 Convergence Analysis

To obtain convergent results of the linear θ-method with the Grünwald-Letnikov operator, it is essential to introduce the following lemma.

Lemma 5 ([6]) *For the generalized initial-boundary value problem of partial differential equations, the proposed numerical method is convergent if it is stable and consistent* (*see*, *e.g.*, [35]), *which is commonly known as Lax equivalence Theorem.*

Applying Lemma 1 to the numerical discretizations (11), we have

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t \theta u_t(x_i, y_j, t_{k+1}) + \Delta t(1-\theta) u_t(x_i, y_j, t_k) + \Delta t O\left(h_x + h_y\right), \tag{40}$$

where $u_t(x_i, y_j, t_{k+1})$ and $U(x_i, y_j, t_{k+1})$ are specifically extended to three orders by utilizing *Taylor* series expansion, then $\tilde{R}_{i,j}^k(\theta)$ can be consequently simplified as follows:

$$
\begin{aligned}
\tilde{R}_{i,j}^{k}(\theta) &= U(x_i, y_j, t_{k+1}) - u_{i,j}^{k+1} \\
&= \left(\frac{1}{2} - \theta\right)\left[\frac{\partial^2 u}{\partial t^2}\right]_{i,j}^{k}(\Delta t)^2 + \left(\frac{1}{6} - \frac{\theta}{2}\right)\left[\frac{\partial^3 u}{\partial t^3}\right]_{i,j}^{k}(\Delta t)^3 \\
&\quad + O\big((\Delta t)^4\big) + \Delta t O\big(h_x + h_y\big).
\end{aligned}
$$

Basic calculations divide $\tilde{R}_{i,j}^{k}(\theta)$ into two cases with different values of θ, which are supported to obtain the following result.

Theorem 5 *Suppose that the assumptions in Theorem 3 and* (A1)–(A2) *hold, the linear θ-method with the Grünwald-Letnikov operator is convergent. Besides, there exists a positive constant $\widehat{C} > 0$ independent on any step-sizes, such that*

$$
\|\epsilon^k\|_\infty \leqslant \begin{cases} \widehat{C}\big((\Delta t)^2 + h_x + h_y\big), & \theta = 1/2, \\ \widehat{C}(\Delta t + h_x + h_y), & \theta \in [0, 1/2) \cup (1/2, 1]. \end{cases} \tag{41}
$$

Proof Denote $u(x_i, y_j, t_k)$ as analytic solutions of Problem (1), then we define $\epsilon_{i,j}^{k} = u(x_i, y_j, t_k) - u_{i,j}^{k}$, and

$$
\epsilon^k = \left(\epsilon_{1,1}^{k}, \cdots, \epsilon_{M_x-1,1}^{k}, \epsilon_{1,2}^{k}, \cdots, \epsilon_{M_x-1,2}^{k}, \cdots, \epsilon_{1,M_y-1}^{k}, \cdots, \epsilon_{M_x-1,M_y-1}^{k}\right)^{\mathrm{T}}.
$$

In the subsequent stage, the Lipschitz condition related to the nonlinear reaction term $\eta_{i,j}^{k} = f\left(x_i, y_j, t_k, u(x_i, y_j, t_{k-m})\right) - f\left(x_i, y_j, t_k, u_{i,j}^{k-m}\right)$ should be guaranteed as

$$
\left|\eta_{i,j}^{k}\right| \leqslant L_2\left|u(x_i, y_j, t_{k-m}) - u_{i,j}^{k-m}\right|, L_2 \in \mathbb{R}^*, \tag{42}
$$

where the same condition applies to

$$
\eta^k = \left(\eta_{1,1}^{k}, \cdots, \eta_{M_x-1,1}^{k}, \eta_{1,2}^{k}, \cdots, \eta_{M_x-1,2}^{k}, \cdots, \eta_{1,M_y-1}^{k}, \cdots, \eta_{M_x-1,M_y-1}^{k}\right)^{\mathrm{T}}.
$$

Similar to (33), we will focus more on

$$
\begin{aligned}
\|\epsilon^{k+1}\|_\infty \leqslant{} & \left\|\big(I + \theta\big(\tilde{a}_x\tilde{S} + \tilde{b}_y\tilde{T}\big)\big)^{-1}\big(I - (1-\theta)\big(\tilde{a}_x\tilde{S} + \tilde{b}_y\tilde{T}\big)\big)\right\|_\infty \|\epsilon^k\|_\infty \\
& + \left\|\big(I + \theta\big(\tilde{a}_x\tilde{S} + \tilde{b}_y\tilde{T}\big)\big)^{-1}\right\|_\infty \Delta t\big(\theta\|\eta^{k+1}\|_\infty + (1-\theta)\|\eta^k\|_\infty + \|R^k\|_\infty\big),
\end{aligned}
$$

where the following error term can be simplified:

$$
\Delta t\theta\|\eta^{k+1}\|_\infty + \Delta t(1-\theta)\|\eta^k\|_\infty \leqslant L_2\Delta t\theta\|\epsilon^{k+1-m}\|_\infty + L_2\Delta t(1-\theta)\|\epsilon^{k-m}\|_\infty,
$$

and (35) is equally satisfied under various θ with corresponding conditions.

Then we generate the convergence in two manners.

Case 1 When $\theta = 1/2$, the maximum norm of R^k satisfies

$$
\|R^k\|_\infty \leqslant \widehat{C_1}\big((\Delta t)^2 + h_x + h_y\big), \tag{43}
$$

where $\widehat{C_1}$ is a positive constant.

Suppose that $\hat{e}^k = \max_{0\leqslant i\leqslant k}\left\{\|\epsilon^i\|_\infty\right\}$, we clearly notice that

$$\hat{e}^0 = 0, \hat{e}^0 \leqslant \hat{e}^1 \leqslant \cdots \leqslant \hat{e}^{k-1} \leqslant \hat{e}^k. \tag{44}$$

Therefore, the error $\|\epsilon^{k+1}\|_\infty$ can be fully improved as follows:

$$\begin{aligned}
\hat{e}^{k+1} \leqslant\; & \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\left(I - \tilde{a}_x(1-\theta)\tilde{S} - \tilde{b}_y(1-\theta)\tilde{T}\right) \right\|_\infty \hat{e}^k \\
& + \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1} \right\|_\infty \frac{L_2}{2}\Delta t\left(\hat{e}^{k+1-m} + \hat{e}^{k-m}\right) \\
& + \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1} \right\|_\infty \widehat{C_1}\Delta t\left((\Delta t)^2 + h_x + h_y\right) \\
\leqslant\; & \left(1 + L_2\Delta t\right)\hat{e}^k + \widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right).
\end{aligned}$$

Denote $\tilde{\xi} \triangleq 1 + L_2\Delta t$, then we have

$$\begin{aligned}
\hat{e}^{k+1} & \leqslant \tilde{\xi}\hat{e}^k + \widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right) \\
& \leqslant \tilde{\xi}^2\hat{e}^{k-1} + \tilde{\xi}\widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right) + \widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right) \\
& \cdots \\
& \leqslant \tilde{\xi}^{k+1}\hat{e}^0 + \widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right)\left(\tilde{\xi}^k + \tilde{\xi}^{k-1} + \cdots + \tilde{\xi} + 1\right) \\
& \leqslant \tilde{\xi}^{k+1}\hat{e}^0 + \widehat{C_1}\left((\Delta t)^3 + \Delta t\left(h_x + h_y\right)\right)\left(\frac{\tilde{\xi}^{k+1} - 1}{\tilde{\xi} - 1}\right) \\
& = \tilde{\xi}^{k+1}\hat{e}^0 + \widehat{C_1}\left((\Delta t)^2 + h_x + h_y\right)\Delta t\left(\frac{\tilde{\xi}^{k+1} - 1}{L_2\Delta t}\right) \\
& \leqslant \tilde{\xi}^{k+1}\left(\frac{\widehat{C_1}}{L_2}\left((\Delta t)^2 + h_x + h_y\right) + \hat{e}^0\right).
\end{aligned} \tag{45}$$

Similar to (38) and assisted by $\hat{e}^0 = 0$ from (44), the above-mentioned iterations (45) result in the following relations by applying discretized Grönwall inequalities:

$$\begin{aligned}
\hat{e}^{k+1} & \leqslant \frac{\widehat{C_1}}{L_2}\left((\Delta t)^2 + h_x + h_y\right)\left(1 + L_2\Delta t\right)^{k+1} \\
& \leqslant \frac{\widehat{C_1}}{L_2}\exp\left(L_2 T\right)\left((\Delta t)^2 + h_x + h_y\right) \leqslant \hat{C}\left((\Delta t)^2 + h_x + h_y\right),
\end{aligned} \tag{46}$$

which indicates that the numerical method defined by (20) is convergent.

Case 2 When $\theta \in [0, 1/2) \cup (1/2, 1]$, the maximum norm of R^k satisfies

$$\|R^k\|_\infty \leqslant \widehat{C_1}\left(\Delta t + h_x + h_y\right), \tag{47}$$

where $\widehat{C_1}$ is a positive constant. We can clearly illustrate $\hat{e}^{k+1}$ satisfies

$$\begin{aligned}
\hat{e}^{k+1} \leqslant\; & \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1}\left(I - \tilde{a}_x(1-\theta)\tilde{S} - \tilde{b}_y(1-\theta)\tilde{T}\right) \right\|_\infty \hat{e}^k \\
& + \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1} \right\|_\infty L_2\Delta t\left(\theta\hat{e}^{k+1-m} + (1-\theta)\hat{e}^{k-m}\right) \\
& + \left\| \left(I + \tilde{a}_x\theta\tilde{S} + \tilde{b}_y\theta\tilde{T}\right)^{-1} \right\|_\infty \widehat{C_1}\left((\Delta t)^2 + \Delta t\left(h_x + h_y\right)\right) \\
\leqslant\; & \left(1 + L_2\Delta t\right)\hat{e}^k + \widehat{C_1}\left((\Delta t)^2 + \Delta t\left(h_x + h_y\right)\right),
\end{aligned} \tag{48}$$

therefore, Inequalities (48) give

$$
\begin{aligned}
\hat{e}^{k+1} &\leqslant \left(1+L_2\Delta t\right)\hat{e}^k + \widehat{C_1}\left((\Delta t)^2+\Delta t\left(h_x+h_y\right)\right)\\
&\leqslant \frac{\widehat{C_1}}{L_2}(\Delta t+h_x+h_y)\left(1+L_2\Delta t\right)^{k+1} \leqslant \frac{\widehat{C_1}}{L_2}\exp\left(L_2T\right)\left(\Delta t+h_x+h_y\right)\\
&\leqslant \widehat{C}\left(\Delta t+h_x+h_y\right).
\end{aligned}
$$

To sum up, the convergence of the proposed numerical method is evidently proved.

4 Numerical Experiments

In this section, we will implement an example to demonstrate the effectiveness of theoretical conclusions. Consider the following equations:

$$
\begin{cases}
u_t(x,y,t) = 3.75\partial_x^{\alpha}u(x,y,t) + 2\times 10^{-4}\partial_y^{\beta}u(x,y,t)\\
\qquad\qquad + u(x,y,t-1)(1-u(x,y,t-1)) + f(x,y,t), t\in(0,T],\\
u(0,y,t)=0, \dfrac{\partial}{\partial x}u(1,y,t)=0, y\in[0,1],\\
u(x,0,t)=0, \dfrac{\partial}{\partial y}u(x,1,t)=0, x\in[0,1],\\
u(x,y,t)=\mathrm{e}^{-t}x^4(x-1)^4y^4(y-1)^4, t\in[-1,0], 0\leqslant x,y\leqslant 1.
\end{cases}
\tag{49}
$$

Moreover, $f(x, y, t)$ is relatively defined by

$$
\begin{aligned}
f(x,y,t) = -10\Bigg(&\mathrm{e}^{-t}\tilde{g}(x,y) - \kappa_2\frac{\mathrm{e}^{-t}y^4(1-y)^4}{2\cos(\alpha\pi/2)}\tilde{f}^{\alpha}(x,t) - \gamma_2\frac{\mathrm{e}^{-t}x^4(1-x)^4}{2\cos(\beta\pi/2)}\tilde{f}^{\beta}(y,t)\Bigg)\\
&- 10\mathrm{e}^{-(t-\tau)}\tilde{g}(x,y)\left(1-10\mathrm{e}^{-(t-\tau)}\tilde{g}(x,y)\right),
\end{aligned}
$$

where $\tilde{g}(x,y)=x^4(1-x)^4y^4(1-y)^4$, and $\tilde{f}^{\alpha}(x,t)$,$\tilde{f}^{\beta}(y,t)$ are denoted as follows:

$$
\begin{aligned}
\tilde{f}^{\alpha}(x,t) = &(6-\alpha)(5-\alpha)\frac{\Gamma(5)}{\Gamma(7-\alpha)}\left(x^{4-\alpha}+(1-x)^{4-\alpha}\right)\\
&-4(7-\alpha)(6-\alpha)\frac{\Gamma(6)}{\Gamma(8-\alpha)}\left(x^{5-\alpha}+(1-x)^{5-\alpha}\right)\\
&+6(8-\alpha)(7-\alpha)\frac{\Gamma(7)}{\Gamma(9-\alpha)}\left(x^{6-\alpha}+(1-x)^{6-\alpha}\right)\\
&-4(9-\alpha)(8-\alpha)\frac{\Gamma(8)}{\Gamma(10-\alpha)}\left(x^{7-\alpha}+(1-x)^{7-\alpha}\right)\\
&+(10-\alpha)(9-\alpha)\frac{\Gamma(9)}{\Gamma(11-\alpha)}\left(x^{8-\alpha}+(1-x)^{8-\alpha}\right),
\end{aligned}
\tag{50a}
$$

$$
\begin{aligned}
\tilde{f}^{\beta}(y,t) = {} & (6-\beta)(5-\beta)\frac{\Gamma(5)}{\Gamma(7-\beta)}\left(y^{4-\beta}+(1-y)^{4-\beta}\right) \\
& -4(7-\beta)(6-\beta)\frac{\Gamma(6)}{\Gamma(8-\beta)}\left(y^{5-\beta}+(1-y)^{5-\beta}\right) \\
& +6(8-\beta)(7-\beta)\frac{\Gamma(7)}{\Gamma(9-\beta)}\left(y^{6-\beta}+(1-y)^{6-\beta}\right) \\
& -4(9-\beta)(8-\beta)\frac{\Gamma(8)}{\Gamma(10-\beta)}\left(y^{7-\beta}+(1-y)^{7-\beta}\right) \\
& +(10-\beta)(9-\beta)\frac{\Gamma(9)}{\Gamma(11-\beta)}\left(y^{8-\beta}+(1-y)^{8-\beta}\right).
\end{aligned} \tag{50b}
$$

Here, the analytic solution of Problem (49) is $u(x,y,t) = 10\mathrm{e}^{-t}x^4(1-x)^4y^4(1-y)^4$. Following that, we will verify the consistency between theoretical and numerical conclusions under various parameters.

Stability Tests. To illustrate Theorem 3 regarding the conditional stability, we primarily take $\theta = 1/4$ into consideration, accompanied with fixed $h_x = h_y = 1/10$ and $\alpha = 1.8, \beta = 1.6$. By simple calculations, it is worth noting that the stability of Problem (49) is limitedly satisfied when $m \geqslant 136$, mainly relies on the maximum eigenvalue $\lambda = 3.633\,9$.

Thus, let $m = 300$ as an appropriate situation conformed with (28), we directly observe that the numerical solutions are stable in Fig. 1. Furthermore, paying attention to Fig. 2, it is surprising that though the corresponding solutions are stable with several tiny spatial-intervals, the numerical stability is still inexplicable due to these enormous values. Particularly, we proceed to consider $\theta = 3/4$. These subsequent pictorial results with different step-sizes are shown in Figs. 3 and 4, which get a better understanding of the unconditional stability.

Convergence Tests. At the very beginning, we define the proposed errors between analytic and numerical solutions

$$
E_\infty(h_x,h_y,\Delta t) = \max_{\substack{1\leqslant i\leqslant M_x, 1\leqslant j\leqslant M_y,\\ 1\leqslant k\leqslant N}} \left|u(x_i,y_j,t_k) - u_{i,j}^k\right|. \tag{51}
$$

Meanwhile, two kinds of orders concerning convergence in the L_∞-norm with regard to temporal and spatial directions are, respectively, defined by

$$
\mathrm{Ord1}_\infty = \log_2\left(\frac{E_\infty(2h_x,2h_y,\Delta t)}{E_\infty(h_x,h_y,\Delta t)}\right), \mathrm{Ord2}_\infty = \log_2\left(\frac{E_\infty(2h_x,2h_y,2\Delta t)}{E_\infty(h_x,h_y,\Delta t)}\right).
$$

We preliminarily explore the convergence along spatial directions involved h_x, h_y. To assure the stability, a tiny temporal step-size $\Delta t = 1\times 10^{-3}$ is necessarily fixed. Based on couples of numerical simulations, the error estimations and the orders of convergence are enumerated in Table 2 with multiple coefficients. By briefly glancing at this tabulation, we can discern that the numerical method in Sect. 3 has a better convergence, since $\mathrm{Ord1}_\infty$ approximates to 1 when $\theta = 0.5, 0.75, 1$.

Beyond that, the global convergence along spatial and temporal directions will be verified. Corresponding error estimations and the orders of convergence link efficiently with multiple computations are verified in Table 2 as well. What is notable, is that $\mathrm{Ord2}_\infty$

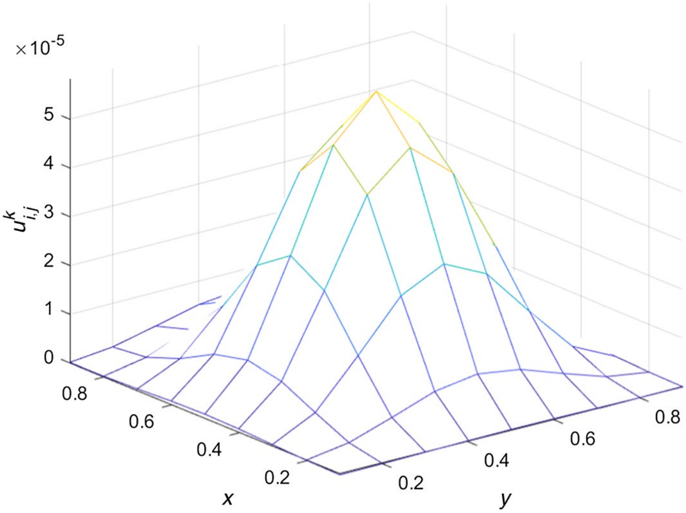

Fig. 1 Numerical solutions of Problem (49) when $m = 300, T = 1, \theta = 0.25$

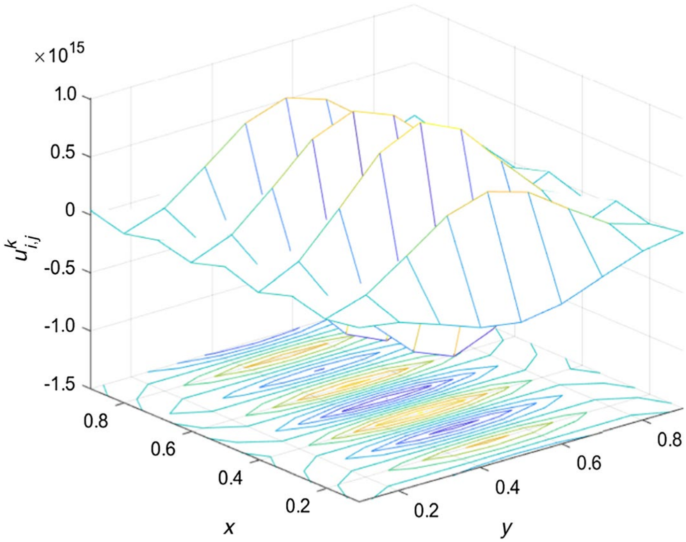

Fig. 2 Numerical solutions of Problem (49) when $m = 100, T = 1, \theta = 0.25$

approaches to 2 with $\theta = 0.5$ and approximates to 1 with $\theta = 0.75, 1$, which validates Theorem 5 owing to their good convergence.

Furthermore, we will investigate the interaction effect of fractional derivatives α and β, respectively. To achieve this, it is necessary to reduce the two-dimensional problem to a lower order.

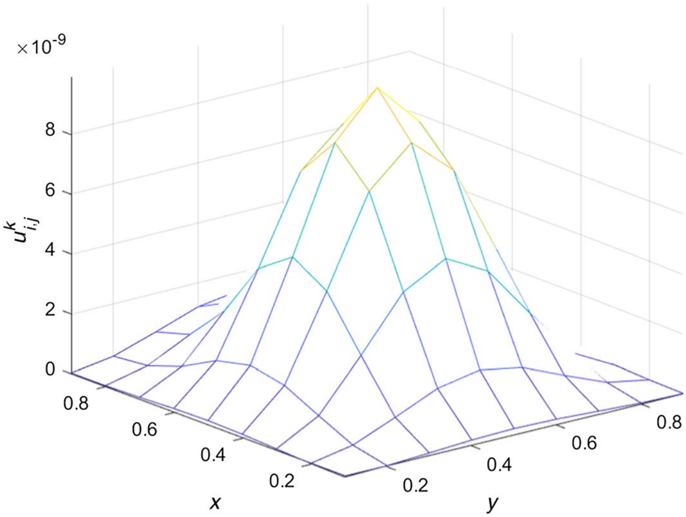

Fig. 3 Numerical solutions of Problem (49) when $m = 100, T = 10, \theta = 0.75$

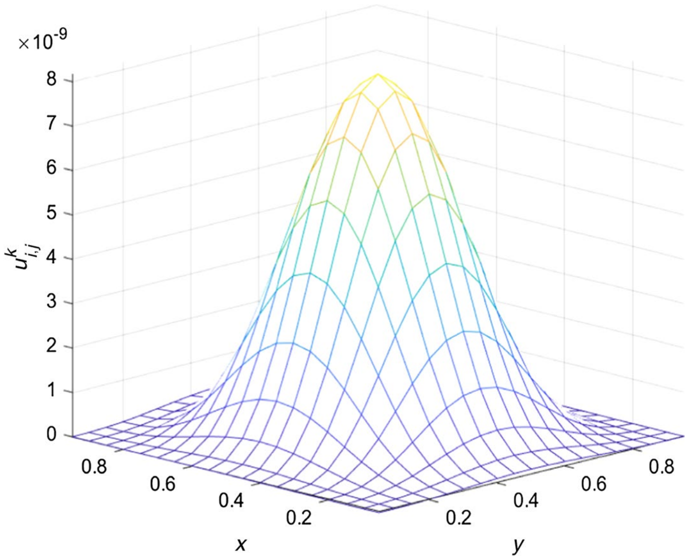

Fig. 4 Numerical solutions of Problem (49) when $m = 400, T = 10, \theta = 0.75$

Tests of the Interaction Impact of Fractional Derivatives. In the preliminary stage, we should select and fix an appropriate point to better explore the impact of space-fractional derivatives α, β.

Denote $h_x^2 = h_y^2 = \Delta t$ when $h_x = 1/20, \theta = 0.75$. According to the information provided in Fig. 5, the computational simulations are depicted for fixed β and various α. Compared with the analytic solutions, the order of space-fractional derivative α has a profound effect

Table 2 The errors and orders of convergence with different parameters when $T = 8$

Various numerical errors computed by the linear θ-method with the Grünwald-Letnikov operator							
$h_x = h_y$	Δt	$\theta = 0.5$		$\theta = 0.75$		$\theta = 1$	
when $\alpha = \beta = 1.5$		E_∞	$\mathrm{Ord1}_\infty$	E_∞	$\mathrm{Ord1}_\infty$	E_∞	$\mathrm{Ord1}_\infty$
1/5	1/1 000	3.192 1E−08		3.192 2E−08		2.525 3E−07	
1/10		1.703 8E−08	0.905 731	1.703 9E−08	0.905 708	1.344 6E−07	0.909 264
1/20		8.683 2E−09	0.972 496	8.683 6E−09	0.972 481	6.234 8E−08	1.108 752
1/40		4.368 3E−09	0.991 139	4.368 6E−09	0.991 136	3.102 0E−08	1.007 133
$h_x = h_y$	Δt	$\theta = 0.5$		$\theta = 0.75$		$\theta = 1$	
when $\alpha = 1.8, \beta = 1.6$		E_∞	$\mathrm{Ord2}_\infty$	E_∞	$\mathrm{Ord2}_\infty$	E_∞	$\mathrm{Ord2}_\infty$
1/5	1/25	3.487 1E−08		3.487 0E−08		3.046 9E−08	
1/10	1/100	8.975 2E−09	1.957 997	1.597 3E−08	1.126 321	1.458 1E−08	1.063 254
1/20	1/400	2.416 6E−09	1.892 972	8.082 1E−09	0.982 857	7.063 6E−09	1.045 633
1/40	1/1 600	6.247 9E−10	1.951 533	3.695 1E−09	1.129 107	3.325 1E−09	1.087 000

on the numerical calculations. To be specific, the numerical solutions completely approximate to the analytic solutions under the situation $\alpha = \beta$.

Additionally, the simulative computations involved proposed numerical methods for fixed α and various β are presented in Fig. 6. Nevertheless, three groups of numerical solutions in this graph are roughly equal, which indicates that different values of β have slight impact on numerical analysis. What gives rise to this phenomenon is likely to depend on the tiny value of the diffusion parameter γ_2.

Remark 1 When considering space-fractional derivatives are discretized by the centered difference operator with second-order accuracy novelly applied in [7, 14, 20], relevant numerical tests are acquired by the linear θ-method, where the computing procedure is simplified.

Denote $C \triangleq$ Centered Difference Operator and $G \triangleq$ Grünwald-Letnikov Operator.

In accordance with the following information compared with Figs. 5 and 7, we can detect that similar observations of influence caused by fractional derivatives are presented under the same situations mentioned above. However, if we look closer at the fluctuation reflected in Fig. 8, the feasibility associated with the Grünwald-Letnikov operator is much steadier in terms of smoothness and error limitations.

5 Concluding Remarks

In summary, we have studied the linear θ-method with the Grünwald-Letnikov operator based on the unconditional stability of analytic solutions for generalized delayed space-fractional Fisher equations in two dimensions. The ADIM has been beneficially introduced to simplify the calculations and matrices' integrations. Then we successively demonstrated the numerical scheme is conditionally stable with $\theta \in [0, 1/2)$ and its unconditional

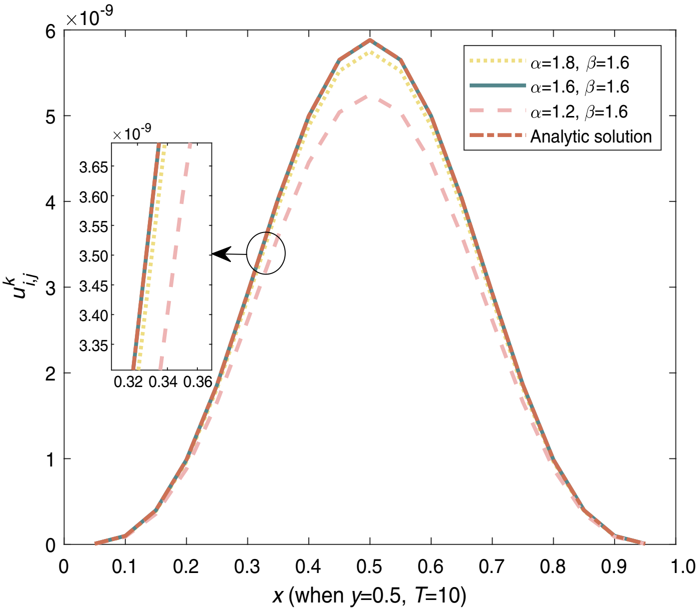

Fig. 5 Numerical solutions and analytic solutions of Problem (49) with different α when $\beta = 1.6, y = 0.5, T = 10$

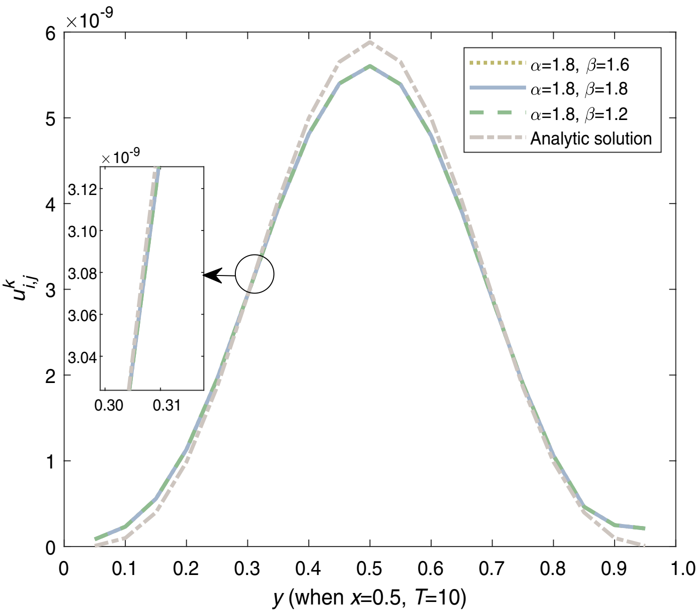

Fig. 6 Numerical solutions and analytic solutions of Problem (49) with different β when $\alpha = 1.8, x = 0.5, T = 10$

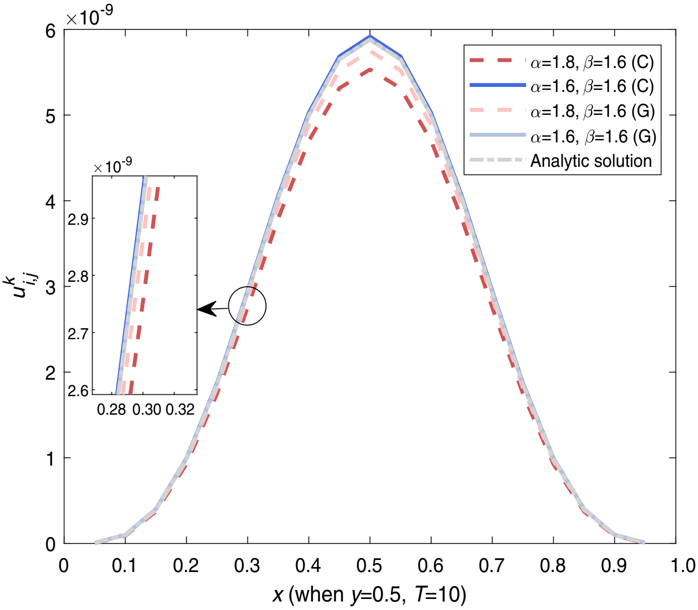

Fig. 7 Numerical computations of Problem (49) with two different methods and various α when $\beta = 1.6, y = 0.5, T = 10$

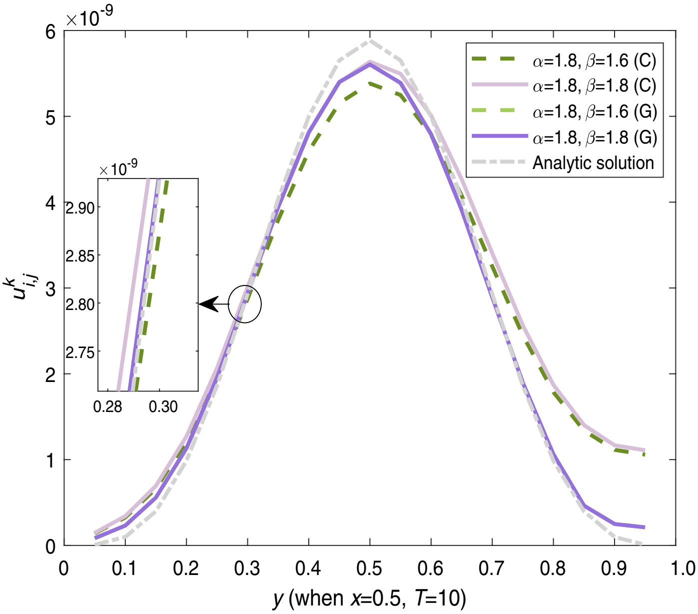

Fig. 8 Numerical computations of Problem (49) with two different methods and various β when $\alpha = 1.8, x = 0.5, T = 10$

stability with $\theta \in [1/2, 1]$. Under those comprehensive investigations, the convergence has been discussed by utilizing Taylor expansions and error estimations with different θ.

Numerical experiments showed that the proposed computing method has a rational stability and convergence consistent with theoretical results, especially the interaction impacts of space-fractional derivatives calculated by linear θ-methods with two harmonic operators have been further interpreted. A notable thing is that the Grünwald-Letnikov operator with first-order precision displays a surprisingly better simulation than other higher-order counterparts. Due to the complicated as well as generalized cases resulted by practical applications, further complicating the picture is deeper research of seeking requisite conditions for more steady numerical results simulated by difference operators with the higher accuracy, and we will consider multiple delayed dimensional problems with nonsmooth solutions in the future.

Acknowledgements The authors wish to gratitude the referees for their valuable advice and comments, which assisted in improving overall presentations of the manuscript.

Funding This work is supported by the National Natural Science Foundation of China (No. 11201084).

Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no competing interests.

Informed Consent On behalf of the authors, Dr. Qi Wang shall be communicating the manuscript.

References

1. Achouri, T., Ayadi, M., Habbal, A., Yahyaoui, B.: Numerical analysis for the two-dimensional Fisher-Kolmogorov-Petrovski-Piskunov equation with mixed boundary condition. J. Appl. Math. Comput. **68**(6), 3589–3614 (2021)
2. Ahmad, S., Ullah, A., Partohaghighi, M., Saifullah, S., Akgul, A., Jarad, F.: Oscillatory and complex behaviour of Caputo-Fabrizio fractional order HIV-1 infection model. AIMS Math. **7**(3), 4778–4792 (2022)
3. Arfaoui, H., Makhlouf, A.B.: Some results for a class of two-dimensional fractional hyperbolic differential systems with time delay. J. Appl. Math. Comput. **68**(4), 2389–2405 (2021)
4. Atangana, A.: On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation. Appl. Math. Comput. **273**, 948–956 (2016)
5. Blanco-Cocom, L., Avila-Vales, E.: Convergence and stability analysis of the θ-method for delayed diffusion mathematical models. Appl. Math. Comput. **231**, 16–25 (2014)
6. Butzer, P.L., Diekmeis, W., Jansen, H., Nessel, R.J.: Alternative forms with orders of the Lax equivalence theorem in Banach spaces. Computing **17**(4), 335–342 (1977)
7. Celik, C., Duman, M.: Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. J. Comput. Phys. **231**(4), 1743–1750 (2012)
8. Chen, S., Liu, F., Turner, I., Anh, V.: An implicit numerical method for the two-dimensional fractional percolation equation. Appl. Math. Comput. **219**(9), 4322–4331 (2013)
9. Corti, M., Antonietti, P.F., Bonizzoni, F., Dede, L., Quarteroni, A.: Discontinuous Galerkin methods for Fisher-Kolmogorov equation with application to α-Synuclein spreading in Parkinson's disease. Comput. Methods Appl. Mech. Eng. **417**, 116450 (2023)
10. Dehghan, M., Abbaszadeh, M.: A finite difference/finite element technique with error estimate for space fractional tempered diffusion-wave equation. Comput. Math. Appl. **75**(8), 2903–2914 (2018)
11. Demir, A., Bayrak, M.A., Ozbilge, E.: An approximate solution of the time-fractional Fisher equation with small delay by residual power series method. Math. Probl. Eng. **2018**, 1–8 (2018)

12. El-Danaf, T.S., Hadhoud, A.R.: Computational method for solving space fractional Fisher's nonlinear equation. Math. Method. Appl. Sci. **37**(5), 657–662 (2013)
13. Garmanjani, G., Cavoretto, R., Esmaeilbeigi, M.: A RBF partition of unity collocation method based on finite difference for initial-boundary value problems. Comput. Math. Appl. **75**(11), 4066–4090 (2018)
14. Hao, Z., Zhang, Z., Du, R.: Fractional centered difference scheme for high-dimensional integral fractional Laplacian. J. Comput. Phys. **424**, 109851–109868 (2021)
15. Hendy, A.S., Zaky, M.A., De Staelen, R.H.: A general framework for the numerical analysis of high-order finite difference solvers for nonlinear multi-term time-space fractional partial differential equations with time delay. Appl. Numer. Math. **169**, 108–121 (2021)
16. Horn, R., Johnson, C.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
17. Ilati, M.: Analysis and application of the interpolating element-free Galerkin method for extended Fisher-Kolmogorov equation which arises in brain tumor dynamics modeling. Numer. Algorithms **85**(2), 485–502 (2019)
18. Izadi, M., Srivastava, H.M.: An optimized second order numerical scheme applied to the non-linear Fisher's reaction-diffusion equation. J. Interdiscip. Math. **25**(2), 471–492 (2022)
19. Kenkre, V.: Results from variants of the Fisher equation in the study of epidemics and bacteria. Phys. A. **342**(1/2), 242–248 (2004)
20. Kwak, D.Y., Kwon, H.J., Lee, S.: Multigrid algorithm for cell centered finite difference on triangular meshes. Appl. Math. Comput. **105**, 77–85 (1999)
21. Liu, F., Chen, S., Turner, I., Burrage, K., Anh, V.: Numerical simulation for two-dimensional Riesz space fractional diffusion equations with a nonlinear reaction term. Open. Phys. **11**(10), 1221–1232 (2013)
22. Liu, M.Z., Spuker, M.N.: The stability of the θ-methods in the numerical solution of delay differential equations. IMA. J. Numer. Anal. **10**, 31–48 (1990)
23. Liu, Z., Zeng, S., Bai, Y.: Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications. Fract. Calc. Appl. Anal. **19**(1), 188–211 (2016)
24. Macias-Diaz, J.: A bounded numerical method for approximating a hyperbolic and convective generalization of Fisher's model with nonlinear damping. Appl. Math. Lett. **25**(6), 946–951 (2012)
25. Majeed, A., Kamran, M., Abbas, M., Singh, J.: An efficient numerical technique for solving time-fractional generalized Fisher's equation. Front. Phys. **8**, 293 (2020)
26. Meerschaert, M.M., Scheffler, H.P., Tadjeran, C.: Finite difference methods for two-dimensional fractional dispersion equation. J. Comput. Phys. **211**, 249–261 (2006)
27. Oruç, O.: An efficient wavelet collocation method for nonlinear two-space dimensional Fisher-Kolmogorov-Petrovsky-Piscounov equation and two-space dimensional extended Fisher-Kolmogorov equation. Eng. Comput. **36**(3), 839–856 (2019)
28. Ozkose, F., Yavuz, M., Senel, M.T., Habbireeh, R.: Fractional order modelling of omicron SARS-CoV-2 variant containing heart attack effect using real data from the United Kingdom. Chaos Solitons Fractals **157**, 111954–111978 (2022)
29. Pan, X., Shu, H., Wang, L., Wang, X.S.: Dirichlet problem for a delayed diffusive hematopoiesis model. Nonlinear Anal. Real. World Appl. **48**, 493–516 (2019)
30. Rashid, S., Kubra, K.T., Sultana, S., Agarwal, P., Osman, M.: An approximate analytical view of physical and biological models in the setting of Caputo operator via Elzaki transform decomposition method. J. Comput. Appl. Math. **413**, 114378–114401 (2022)
31. Roul, P., Rohil, V.: A high order numerical technique and its analysis for nonlinear generalized Fisher's equation. J. Comput. Appl. Math. **406**, 114047–114065 (2022)
32. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Transl. from Russian. Gordon and Breach, New York (1993)
33. Shakeel, M., Hussain, I., Ahmad, H., Ahmad, I., Thounthong, P., Zhang, Y.F.: Meshless technique for the solution of time-fractional partial differential equations having real-world applications. J. Funct. Spaces **2020**, 1–17 (2020)
34. Sun, N., Fang, J.: Propagation dynamics of Fisher-KPP equation with time delay and free boundaries. Calc. Var. Partial Differential Equations **58**, 148–186 (2019)
35. Thomas, J.: Numerical Partial Differential Equations: Finite Difference Methods. Springer, Berlin (1998)
36. Travis, C.C., Webb, G.F.: Existence and stability for partial functional differential equations. Trans. Am. Math. Soc. **200**, 395–418 (1974)
37. Wang, F., Khan, M.N., Ahmad, I., Ahmad, H., Abu-Zinadah, H., Chu, Y.: Numerical solution of travelling waves in chemical kinetics: time-fractional Fishers equations. Fractals **30**(2), 2240051–2240062 (2022)

38. Wu, F., Li, D., Wen, J., Duan, J.: Stability and convergence of compact finite difference method for parabolic problems with delay. Appl. Math. Comput. **322**(9), 129–139 (2018)
39. Yang, S.: Numerical simulation for the two-dimensional and three-dimensional Riesz space fractional diffusion equations with delay and a nonlinear reaction term. Int. J. Comput. Math. **96**(10), 1957–1978 (2018)
40. Zaky, M.A., Hendy, A.S., Macias-Diaz, J.E.: Semi-implicit Galerkin-Legendre spectral schemes for nonlinear time-space fractional diffusion-reaction equations with smooth and nonsmooth solutions. J. Sci. Comput. **82**, 13–40 (2020)
41. Zhang, J., Wei, P., Wang, M.: The investigation into the exact solutions of the generalized time-delayed Burgers-Fisher equation with positive fractional power terms. Appl. Math. Model. **36**(5), 2192–2196 (2012)
42. Zhang, Q., Li, T.: Asymptotic stability of compact and linear θ-methods for space fractional delay generalized diffusion equation. J. Sci. Comput. **81**(3), 2413–2446 (2019)
43. Zhang, T., Li, Y.: Exponential Euler scheme of multi-delay Caputo-Fabrizio fractional-order differential equations. Appl. Math. Lett. **124**, 107709–107717 (2022)
44. Zhao, L., Deng, W., Hesthaven, J.S.: Characterization of image spaces of Riemann-Liouville fractional integral operators on Sobolev spaces $W^{m,p}(\Omega)$. Science China Mathematics **64**(12), 2611–2636 (2020)
45. Zhu, X., Nie, Y., Wang, J., Yuan, Z.: A numerical approach for the Riesz space-fractional Fisher' equation in two-dimensions. Int. J. Comput. Math. **94**(2), 296–315 (2015)